MTH 303

Real analysis **Homework 1** **Aug 25, 2014**

1. Prove that there is no rational number whose square root is 12.
2. If r and x are non-zero rational and irrational numbers respectively, prove that $r+x$ and rx are irrational.
3. Consider the following subsets of $\mathbb{Q}$

$$A = \{p \in \mathbb{Q} : p^2 < 2\} \text{ and } B = \{p \in \mathbb{Q} : p^2 > 2\}.$$

   Show that (without using the Archimedian property)
   (a) for any $p \in A$ there is $q \in A$ such that $p < q$
   (b) for any $p \in B$ there is $q \in B$ such that $q < p$.
4. Let $A = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$. Find $\inf A$ and $\sup A$.
5. Let $A = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$. Find $\inf A$ and $\sup A$.
6. Let A and B be two bounded subsets of $\mathbb{R}$. Show that
   (a) $\sup(-A) = -\inf A$
   (b) $\sup(A+B) = \sup A + \sup B$
   (c) $\sup(\alpha A) = \alpha \sup A,\ \alpha > 0$.
7. Let A and B be two nonempty subsets of $\mathbb{R}$ such that $a < b$ for all $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$.
8. Let x be a real number. Show that there exists a unique integer $m \in \mathbb{Z}$ such that

$$m - 1 \leq x < m.$$

9. Let x be a positive real number. Show that there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.
10. Let x and y be two real numbers such that $x < y$, then show that
    (a) there exists an irrational number q such that $x < q < y$
    (b) for every real number $z > 0$ there exist a rational number p and an irrational number q such that

$$x < zp < y \text{ and } x < zq < y.$$

11. Give an explicit definition of a bijective function f between each of the following pairs of sets
    (a) $\mathbb{N}$ and the set of even positive integers
    (b) $\mathbb{N}$ and the set of all odd integers which are greater than 13
    (c) $\mathbb{N}$ and $\mathbb{Z}$

(d) $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$

(e) $(0, 1)$ and $(2, 4)$

(f) $(0, 1)$ and $\mathbb{R}$.

12. Let A be a countable set, and let $B_n,\ n \in \mathbb{N}$, be the set of $n-$tuples $(a_1, a_2, \ldots, a_n)$, where $a_k \in A,\ k = 1, 2, \ldots, n$. Show that B_n is countable.

13. Show that the set of irrational numbers between 0 and 1 is not countable.

14. Prove that the set of algebraic numbers is countable.
Algebraic numbers: Algebraic numbers may be thought of generalizations of rational numbers defined as follows.

    The rational number $x = p/q$ is the solution of a linear equation with integer coefficients; namely, $qx - p = 0$. An algebraic number is a number which occurs as the solution of a polynomial equation with integer coefficients. For example, $\sqrt{2}$ is algebraic, since it is a solution of the quadratic equation $1.x^2 + 0.x - 2 = 0$. The fact that algebraic numbers are countably infinite implies that there must exist numbers which are not the solutions of polynomial equations with integer coefficients. These numbers are called transcendental. For example, π is a transcendental.